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2. S. Das, P. S. Ghosh, Borel amenable action of acylindrically hyperbolic group (under preparation).

Abstract

The study of amenability property for a finitely generated group G was initiated by von Neumann, in 1929, while studying Banach-Tarski paradox. Later it was generalised by several authors. They are Haagerup property (U. Haagerup), weak amenability (U. Haagerup and others), weak Haagerup property (S. Knudby). All these properties are stable under direct product of groups. Further Haagerup property and weak amenability (with $\Lambda_{CH} = 1$) had been proved to be stable under free products, more generally graph products of groups. Here we prove (i) the stability of the Haagerup property under graph product, and (ii) generalise this to the stability of weak Haagerup property with the weak Haagerup constant $\Lambda_{WH} = 1$ under graph products. As a corollary, we prove that free product of weakly Haagerup groups are weakly Haagerup.

In another direction, the notion of amenability was also generalised to amenable actions, by R. Zimmer. Free groups, hyperbolic groups are known to admit amenable action on suitable spaces. N. Ozawa proved amenable action for relatively hyperbolic groups with suitable maximal parabolic subgroups. In this work, we prove Borel amenable action for acylindrically hyperbolic groups, a further generalisations of relatively hyperbolic groups.

Chapter 1

Introduction

Banach and Tarski, in 1920's and 1930's, started investigating, which kinds of group, G, admit a probability measure, that is invariant under the canonical action of G. In 1938, Tarski proved that this kind of measure exists on G if and only if G does not admit a paradoxical decomposition. The study of this property for a general group G was initiated with the work of von Neumann, to characterize paradoxical decomposition of the group.

By a paradoxical decomposition of G we mean to say the existence of a partition $\{A_1, \ldots, A_n, B_1, \ldots, B_m\}$ of G and elements $g_1, \ldots, g_n, h_1, \ldots, h_m$ from G such that

$$G = \cup_{i=1}^{n} g_i \cdot A_i = \cup_{j=1}^{m} h_j \cdot B_j.$$

One can easily show that finite groups, group of integers does not admits such decomposition. However the free group on two generators $\mathbb{F}_2(a, b)$ admits one such decomposition. Notice that $\mathbb{F}_2(a, b) = \{e\} \cup w(a) \cup w(a^{-1}) \cup w(b) \cup w(b^{-1})$, where $w(x)$ is the set of all reduced words that starts with $x \in \mathbb{F}_2(a, b)$, then

$$\mathbb{F}_2(a, b) = w(a) \cup a \cdot w(a^{-1}) = w(b) \cup b \cdot w(b^{-1}).$$

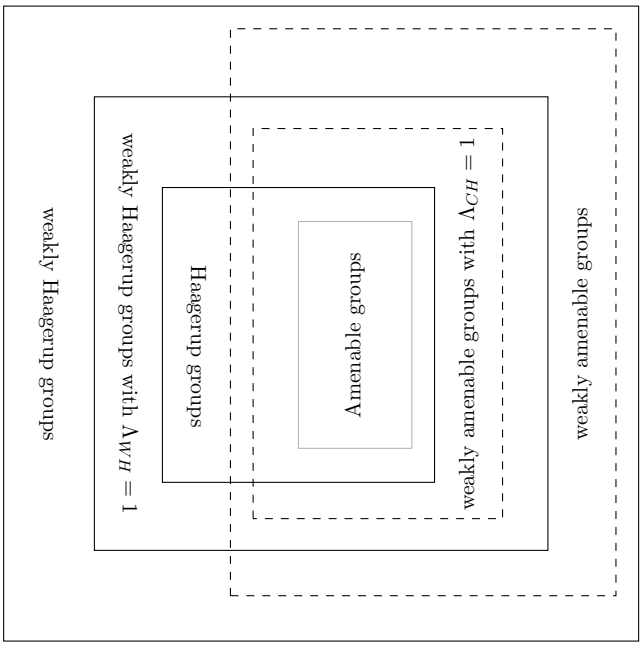
The absence of this decomposition was termed "*amenability*" by M. M. Day in 1950. Since then the topic is well studied and developed and according to Brown and Ozawa [9], there are approximately $10^{10^{10}}$ different equivalent definitions of amenability for a group. The definition most suited for this write up is: a finitely generated group G is amenable if one can define a sequence of finitely supported, positive definite maps on G, converging to 1 point-wise.

Although finite rank free groups are not amenable, in his seminal paper [21] Haagerup showed existence of an approximate identity $\{\phi_n\}$ on the free group $\mathbb{F}_n$, such that they are positive definite, vanishing at infinity (with respect to word metric). Akermann-Walter [2] and Choda [11] called a group to have the *Haagerup property* if it admits a similar approximate identity as in [21]. Clearly amenable groups have Haagerup property. Fundamental group of hyperbolic surfaces, CAT(0)-cubulated groups, groups acting properly on a tree and many others have this property.

One other way amenability can be generalized is called *weak amenability*, where G is required to have approximate identities consisting of finitely supported functions which are,

uniformly bounded in the 'completely bounded'-norm. There is a constant, canonically associated to a weakly amenable group G, obtained as the infimum of the cb-norms of these approximate identities, called the Cowling-Haagerup constant of G and is denoted by $\Lambda_{CH}(G)$.

A further generalisation called the *weak Haagerup Property* interpolates between the Haagerup property and weak amenability. Introduced by Knudby [27], G is called weakly Haagerup if there is an approximate identity of vanishing at infinity maps, uniformly bounded in B_2-norm, on the group. A group G with the weak Haagerup property also comes equipped with a constant $\Lambda_{WH}(G)$, defined similar to Λ_{CH}.


weakly Haagerup groups
weakly Haagerup groups with $\Lambda_{WH} = 1$
Haagerup groups
Amenable groups
weakly amenable groups with $\Lambda_{CH} = 1$
weakly amenable groups

The *graph products of groups* $G(\Gamma)$, defined by Green in her thesis [18], is a novel way to combine a collection $\{G_v\}_{v\in V(\Gamma)}$ of groups parameterised along the vertex set $V(\Gamma)$ of a finite graph Γ. Depending on the nature of the given (finite) graph, a graph product of groups can also be thought of as an interpolation between the free product (which corresponds to a graph without an edge) and the direct product (which corresponds to the complete graph on a given set of vertices) of groups.

One important question in geometric group theory is to know when does a group property remain invariant under various combinations, like the direct product, free product, wreath product, graph product etc., and here we deal with some of them. All the four properties mentioned above are stable under direct product of groups and the properties are also passing down to subgroups. The existence of a free subgroup prevents the group to have amenability (from discussion above). Hence amenability is not preserved under free product of groups. The stability of Haagerup property under free product was done in [10] and that for weak amenability was done in [35] (for the case where $\lambda_{CH} = 1$).

The stability of weak amenability under graph product was done in [34], assuming all

vertex groups have $\Lambda_{CH} = 1$. It was a generalisation of Ricard and Xu's work [35]. Here we will investigate under which conditions Haagerup property and weak Haagerup property shows stability under graph product. Precisely we have the following two results:

Theorem 1.0.1 *[15] Suppose Γ is a finite graph and $\{G_v\}_{v\in V(\Gamma)}$ is a collection of Haagerup groups. Then the graph product $G(\Gamma)$ has Haagerup property.*

Theorem 1.0.2 *[14] Suppose Γ is a finite simplicial graph and $\{G_v\}_{v\in V(\Gamma)}$ is a collection of weakly Haagerup groups with $\Lambda_{WH}(G_v) = 1$, for each v. Then the group $G(\Gamma)$ has the weak Haagerup property with $\Lambda_{WH}(G(\Gamma)) = 1$.*

One more generalisation of amenable groups are those groups which admits amenable actions on some spaces [39] (possibly compact or borel) and are called boundary amenable groups. In geometric group theory a large class of groups are hyperbolic groups. It was proved that these groups admits topological amenable action on the Gromov boundary, which is compact [1]. A more general situation where G acts properly on an exponentially bounded hyperbolic space, was taken care of in [24].

A natural generalisation of hyperbolic groups are relatively hyperbolic groups (groups that are hyperbolic relative to a finite collection of its subgroups). It was proved that these groups admits amenable action on some compact spaces (provided there is "the natural" assumption on maximal parabolic subgroups of the relatively hyperbolic group) [31]. We proved that acylindrically hyperbolic groups (generalisation of R.H. Groups) admits a borel amenable action on some measure space that can be realized as a geometric boundary (in some sense), i.e.

Theorem 1.0.3 *[13] Suppose G is a finitely generated acylindrically hyperbolic group. Then there is a hyperbolic space (not necessary locally finite) on which G acts acylindrically, and the natural action of the group on the boundary of the hyperbolic space (along with a suitable measure) is borel amenable.*

Chapter 2

Preliminaries

Suppose G is a group. From now on G will always be a finitely generated group. Given a function $\phi : G \to \mathbb{C}$, it will induce a kernel $k_\phi : G \times G \to \mathbb{C}$ defined by

$$k_\phi(g, h) := \phi(h^{-1}g).$$

By a positive definite function $\phi : G \to \mathbb{C}$, we mean that for each $n \in \mathbb{N}$, given any $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$ and $\{g_1, g_2, \ldots, g_n\} \subset G$ we have:

$$\sum_{i,j=1}^{n} c_i \overline{c_j} k_\phi(g_i, g_j) = \sum_{i,j=1}^{n} c_i \overline{c_j} \phi(g_j^{-1} g_i) \geq 0.$$

The set $\ell^2 G$ is the space of all square summable functions on G, which will form a Hilbert space with respect to the inner product $\langle f_1, f_2 \rangle := \sum_{g \in G} f_1(g)\overline{f_2(g)}$. $(\ell^2 G)_1$ is the unit sphere of the Hilbert space. An alternative characterization of ϕ being positive definite is ([16]): the existence of an orthogonal representation $\pi : G \to \mathbb{B}(\ell^2 G)$ and $\xi \in (\ell^2 G)_1$ such that,

$$\phi(g) = \langle \pi(g)\xi, \xi \rangle$$

Similarly a ϕ will be called conditionally negative definite (CND) if for any $n \in \mathbb{N}$ the $n \times n$ matrix $\left[K_{ij} = \phi(g_j^{-1} g_i)\right]$ is a conditionally negative definite matrix for arbitrary choice of subsets $\{g_1, g_2, \ldots, g_n\} \subset G$. An equivalent characterization will be the existence of a Hilbert space $\mathcal{H}$ and a map $R : G \to \mathcal{H}$ such that,

$$k_\phi(g, h) = \phi(h^{-1}g) = ||R(g) - R(h)||^2 \tag{2.1}$$

Given any kernel k on G, either positive definite or conditionally negative definite, it will induce a function ϕ_k of same type on G defined by the relation $\phi_k(g) = k(g, e)$, if k is a G-invariant i.e. $k(g, h) = k(h^{-1}g, e)$ for all $(g, h) \in G \times G$. The interaction of positive definite kernels and conditionally negative definite is due to Schoenberg.

Lemma 2.0.1 *([36], Theorem D.11[9])* $k : G \times G \to \mathbb{C}$ *is conditionally negative definite if and only if* $e^{-k} : G \times G \to \mathbb{C}$ *is positive definite.*

A generalisation of positive definite maps are completely bounded maps, which will be used to define the weak Haagerup property, in what follows.

Consider the Hilbert space $\ell^2(G)$. $\mathbb{B}(\ell^2(G))$ is the space of all bounded operators on $\ell^2(G)$. Any element $A \in \mathbb{B}(\ell^2(G))$ can be considered in the matrix form $[A_{g,h}]_{g,h\in G}$, where $A_{g,h} = \langle A\delta_h, \delta_g\rangle$. A kernel $k : G \times G \to \mathbb{C}$, induces a multiplier $m_k : \mathbb{B}(\ell^2(G)) \to \mathbb{B}(\ell^2(G))$ defined by

$$[A_{g,h}]_{g,h\in G} \mapsto [k(g,h)A_{g,h}]_{g,h\in G},$$

called a *Schur multiplier*. Given a function $\phi : G \to \mathbb{C}$, the Schur multiplier m_{k_ϕ} for the induced kernel k_ϕ, is called the *Herz-Schur multiplier* if it is a bounded operator. Further for each $n \in \mathbb{N}$, we can define $M_{n,\phi} : \mathbb{M}_n(\mathbb{B}(\ell^2(G))) \to \mathbb{M}_n(\mathbb{B}(\ell^2(G)))$ by considering the map

$$[a_{ij}] \mapsto [m_{k_\phi}(a_{ij})].$$

Here $\mathbb{M}_n(\mathbb{B}(\ell^2(G)))$ gets its norm by identifying it with $\mathbb{B}((\ell^2(G))^n)$.

Definition 2.0.2 *A function $\phi : G \to \mathbb{C}$ is called completely bounded if there is a $C > 0$ such that the operator-norm of $M_{n,\phi}$, defined above, is bounded by C, for each $n \in \mathbb{N}$. The B_2-norm (alternately the completely bounded norm or the cb-norm) of ϕ is*

$$||\phi||_{B_2} := \sup_{n\in\mathbb{N}} ||M_{n,\phi}||$$

Computing the B_2-norm of a function from the above definition is in general a difficult task. But we have the following equivalent characterisation of a completely bounded function, which is more useful.

Lemma 2.0.3 ([32], Theorem 3) *Let $\phi : G \to \mathbb{C}$ be a function. Then the following are equivalent:*

1. $||\phi||_{B_2} \leq C$,
2. *the operator norm* $||m_{k_\phi}|| \leq C$,
3. *there exist a Hilbert space $\mathcal{H}$ and $\alpha, \beta \in \ell^\infty(G, \mathcal{H})$, such that*

$$k_\phi(g,h) = \langle \alpha(g), \beta(h)\rangle$$

with $\max\{||\alpha||_\infty, ||\beta||_\infty\} \leq \sqrt{C}$

Suppose $\phi(g) := 1$ for all $g \in G$, then it is easy to see $||\phi||_{B_2} = 1$. Further if $\phi(g) := |g|_w$ and G is infinite, then it is not completely bounded as every completely bounded map should be bounded. Moreover any PD function is completely bounded and $||\phi||_{B_2} = |\phi(e)|$.

2.1 Graph product of groups

Suppose Γ is a finite graph, without loops and bigons. $V(\Gamma)$ is vertex set of Γ and $E(\Gamma)$ is the set of edges of Γ. An unordered edge between vertices v and w, is denoted by $[v, w]$. Also let $\{G_v\}_{v\in V(\Gamma)}$ be a fixed family of groups. A graph product $G(\Gamma)$ of $\{G_v\}_{v\in V(\Gamma)}$ is obtained by combining G_v's along Γ.

Definition 2.1.1 *For a finite graph Γ and a family of groups $\{G_v\}_{v\in V(\Gamma)}$, the graph product $G(\Gamma)$ is the quotient $*_{v\in V(\Gamma)}G_v \Big/ \langle\langle\{[G_v, G_w] : [v, w] \in E(\Gamma)\}\rangle\rangle$.*

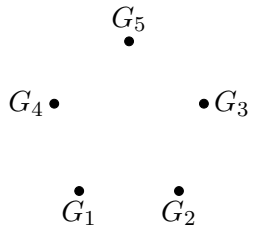


Figure 2.1: Γ is disconnected
$G(\Gamma) = *_{i=1}^{5} G_i$

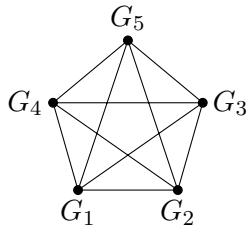


Figure 2.2: Γ is complete
$G(\Gamma) = \oplus_{i=1}^{5} G_i$

The free product and the direct product of groups are the two extreme examples of graph product of groups. One can easily see that if Γ is a graph with n-vertices and no edges then $G(\Gamma)$ is the free product of the vertex groups, and if Γ is a complete graph with n-vertices, then $G(\Gamma)$ is the direct product (see the above figures). Graph product of groups were studied by Green in her thesis [18].

Each element $g \in G(\Gamma)$ can be written in the form $g = g_1 g_2 \dots g_m$ where each $g_i \in G_{v_i}$, for $v_i \in V(\Gamma)$, for some m. If $[v_i, v_{i+1}] \in E(\Gamma)$ then

$$g = g_1 g_2 \dots g_m = g_1 \dots g_{i-1} g_{i+1} g_i g_{i+2} \dots g_m$$

We call the above operation a 'shuffle'. One can perform the following three types of operations on the form $g = g_1 g_2 \dots g_m$:

a) perform shuffle,

b) drop identity element if it appears,

c) multiply g_i and g_{i+1} if both come from G_{v_i}

Remark 2.1.2 *Shuffle does not alter the number of elements. Operations (b) and (c) reduce the number of elements by* 1.

A form $g = g_1 g_2 \dots g_m$, is called *reduced* if the number of elements remains the same after performing the above operations any number of times. This reduced form is the usual normal form in case of free product, which is uniquely determined. The presence of $\mathcal{R}_\Gamma$ relations in a general graph product $G(\Gamma)$ prevents the reduced form to be unique. However, the number of elements in any reduced form of a $g \in G(\Gamma)$, is unique [18, Corollary 3.13]. Therefore it will induce a length function l_r, given by the reduced length $|\cdot|_r$ on $G(\Gamma)$ i.e.

$$\begin{aligned} l_r : G(\Gamma) &\to \mathbb{R} \\ g &\mapsto l_r(g) = |g|_r := m \end{aligned} \tag{2.2}$$

where the reduced length $|g|_r$ of a $g \in G(\Gamma)$ in reduced form $g = g_1 g_2 \dots g_m$, is m.

Remark 2.1.3 *Whenever $g_1 g_2 \dots g_m$ is a reduced form, and $g_i, g_j \in G_v$, then there is some $q \in \mathbb{N}$ and $u \notin st(v)$ such that $i < q < j$, and $g_q \in G_u$, where $st(v) := \{w \in V(\Gamma) : [v, w] \in E(\Gamma)\} \cup \{v\}$.*

Chapter 3

Haagerup Property

The most prominent of all non-amenable groups is the non-abelian free group $\mathbb{F}_n$, on n-generators. In [21], U. Haagerup proved that the non-nuclear reduced group C^*-algebra, $C^*_\lambda(\mathbb{F}_n)$, admits some approximation property, later termed as the Haagerup approximation property or simply *Haagerup property*, by Akermann-Walter [2] and Choda [11] (see definition 3.0.1). The class of groups with Haagerup property is quite large, containing all compact groups (finite groups in the discrete setting), all amenable groups (hence all abelian groups, solvable groups etc.), non-abelian free groups, fundamental groups of closed orientable surfaces etc. An equivalent definition for the groups with Haagerup property is the existence of a proper conditionally negative definite function on such groups. Groups acting properly on a tree [21], or on a $CAT(0)$-cube complex [29], admit such functions, and hence are shown to have Haagerup property. For a good survey on this topic see [10], by Cherix et al.

Yet another way to characterise Haagerup property in a group, often a very useful one, is by showing such groups admit a proper affine isometric action on a Hilbert space. $SO(n,1)$, $SU(n,1)$ admit this kind of action on a Hilbert space, and hence have Haagerup property. Bekka, Cherix and Valette showed that amenable groups also admit such actions, [5]. In this light, Property (T) introduced by D. Kazhdan in [25], can be seen as a strong negation of Haagerup property, which is perhaps the reason behind M. Gromov calling Haagerup groups *a-T-menable*, [19]. For example, as $SL_n(\mathbb{Z})$ has property (T) for $n \geq 3$, they do not have Haagerup property. Other such examples are $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$, $Sp(n,1)$ for $n \geq 2$ etc.

Definition 3.0.1 *A group G is called Haagerup group (or a group with Haagerup property) if there is a sequence of functions $\phi_n : G \to \mathbb{C}$ with the following properties:*

a) $\phi_n(g) \to 1$ point-wise

b) each ϕ_n's vanishes at ∞ i.e. $\phi_n(g) \to 0$ as $|g| \to \infty$,

c) each ϕ_n's are positive definite.

Suppose one has a proper function ϕ on G, which is conditionally negative definite, then form lemma 2.0.1, the sequence $\phi_n = e^{-\frac{\phi}{n}}$ will satisfy all the conditions of definition 3.0.1. The other direction i.e. given a sequence ϕ_n with above properties one can construct a proper conditionally negative definite function ϕ on G, see [10] for the proof of this fact. Therefore we have an equivalent definition:

Definition 3.0.2 *G has Haagerup property if there is a proper, symmetric, conditionally negative definite function on G.*

Whenever G is Haagerup group, we have a proper CND function ϕ, and a map R from G to a Hilbert space $\mathcal{H}$, satisfying equation (2.1). We will call this data, denoted by $(G, \phi, \mathcal{H}, R)$ a Haagerup tuple corresponding to G.

Direct product of two groups with Haagerup property is again a group with Haagerup property. Also, Haagerup property is stable under taking free products and amalgamated products along finite subgroups [10]. Since Haagerup property is stable under both, it seems a natural question to ask whether the Haagerup property remains preserved under any graph product, i.e. to ask: if we take groups $\{G_v\}_{v \in V(\Gamma)}$, one for each vertex in a finite graph Γ and all with Haagerup property, does the graph product satisfy Haagerup property? We answer this question in the positive.

Theorem 3.0.3 *Suppose Γ is a finite graph and $\{G_v\}_{v \in V(\Gamma)}$ is a collection of Haagerup groups. Then the graph product $G(\Gamma)$ has Haagerup property.*

Chapter 4

Weak Haagerup Property

The notion of weak Haagerup property of a group was first conceived by Knudby ([27]), possibly in an attempt to answer the Cowling's conjecture: G is weakly amenable with $\Lambda_{CH} = 1$ if and only if G has Haagerup property (section 1.3.1, [10]). Ozawa and Popa constructed examples of a non-weakly amenable groups in [33], which was later shown to be Haagerup by Cornulier et. al [12]. The other direction of the conjecture remains open. The weak Haagerup property generalizes both the Haagerup property and the weak amenability, giving a common platform to compare both the notions.

Definition 4.0.1 *G is weakly Haagerup if there is a sequence of maps $\phi_n : G \to \mathbb{C}$ with*

1. *ϕ_n vanishes at infinity, for each $n \in \mathbb{N}$,*
2. *$\{\phi_n\}$ is an approximate identity i.e. $\lim_{n\to\infty} \phi_n(g) = 1$ for all $g \in G$,*
3. *there exists a constant $B > 0$ such that $||\phi_n||_{B_2} \leq B$ for each $n \in \mathbb{N}$.*

The weak Haagerup constant, denoted Λ_{WH} for G is the infimum of all such $B > 0$ for which there is a sequence $\{\phi_n\}_n$ satisfying the above properties.

In [27], Knudby provided an equivalent criterion for a weakly Haagerup group with $\Lambda_{WH}(G) = 1$. From proposition 3.1 in [27], a group G is weakly Haagerup with $\Lambda_{WH}(G) = 1$ if and only if there exists a suitable proper map $\phi : G \to \mathbb{R}$ generating a semi-group of approximate identities $\{e^{-\frac{\phi}{n}} : G \to \mathbb{R}\}_n$, vanishing at infinity on G with $||e^{-\frac{\phi}{n}}||_{B_2} \leq 1$. Moreover proposition 4.3 of [27] shows that the above semi-group generator ϕ is given by two functions $R, S : G \to \mathcal{H}$ into some real Hilbert space $\mathcal{H}$ satisfying:

$$\phi(y^{-1}x) = ||R(x) - R(y)||^2 + ||S(x) + S(y)||^2, \quad \text{for all } x, y \in G. \tag{4.1}$$

Moreover without the loss of generality we may assume that $R(1_G) = 0_{\mathcal{H}} \in \mathcal{H}$. From equation 4.1 notice that the image of G under S lies on the sphere of radius $\sqrt{\frac{\phi(1_G)}{4}}$ inside $\mathcal{H}$, i.e. for any $x \in G$:

$$||S(x)||^2 = \frac{\phi(1_G)}{4}$$

Further observe that the first kernel in equation 4.1 is CND. Lemma 2.0.1, tells us that the kernel $G \times G \ni (x, y) \mapsto e^{-||R(x)-R(y)||^2}$ is PD and therefore the B_2-norm of the kernel is 1.

Hence the other part of the kernel $(x, y) \mapsto e^{-\frac{\phi(y^{-1}x)}{n}}$ also has bounded B_2-norm i.e.:

$$||(x, y) \mapsto e^{-\frac{||S(x)+S(y)||^2}{n}}||_{B_2} \leq 1$$

Remark 4.0.2 *By expanding the second term of the right side of equation 4.1, we get the above mentioned expression $\phi = \rho + \tau$: for any $x, y \in G$*

$$\phi(y^{-1}x) = \underbrace{||R(x) - R(y)||^2 + \frac{\phi(1_G)}{2}}_{\rho(x,y)} \quad + \quad \underbrace{2\langle S(x), S(y)\rangle}_{\tau(x,y)}$$

Amenability, Haagerup property and weak amenability are preserved under direct products. In [8], Bozejko and Picardello proved that free product of amenable groups is weakly amenable. Where as it is still an open question whether free product of any two weakly amenable groups is weakly amenable or not. Though as a special case Ricard and Xu had proved that free product of two weakly amenable group with $\Lambda_{CH} = 1$ also has weak amenability [35]. In general, Haagerup property of groups also proved to be stable under taking direct products, free products and amalgamated free products (over finite subgroups) [10]. Knudby in [28] showed that direct product of two weak Haagerup groups is again weakly Haagerup.

For a finite simplicial graph Γ, and a collection $\{G_v\}_{v\in V(\Gamma)}$ of weakly amenable groups with $\Lambda_{CH}(G_v) = 1$ for each vertex $v \in V(\Gamma)$, it was shown by Reckwerdt in [34], that the graph product $G(\Gamma)$ is weakly amenable. We also proved the stability of Haagerup property under graph product in [15], (see also [4]). The main result of this paper establishes stability of the weak Haagerup property under graph product. Note that a group with Haagerup property has $\Lambda_{WH} = 1$, and a weakly amenable group G with $\Lambda_{CH}(G) = 1$ is also weakly Haagerup with $\Lambda_{WH}(G) = 1$. Therefore we have the following theorem.

Theorem 4.0.3 *Suppose Γ is a finite simplicial graph and $\{G_v\}_{v\in V(\Gamma)}$ is a collection of weakly Haagerup groups with $\Lambda_{WH}(G_v) = 1$, for each v. Then the group $G(\Gamma)$ has the weak Haagerup property with $\Lambda_{WH}(G(\Gamma)) = 1$.*

The graph products are in some sense generalisation of free products of groups, so as a consequence of the above theorem we have the following as well.

Corollary 4.0.4 *Suppose A and B are two weakly Haagerup groups with $\Lambda_{WH}(G_v) = 1$. Then the group $G = A * B$ has the weak Haagerup property.*

Chapter 5

Borel Amenable Action

5.1 Acylindrically hyperbolic group

A group G is said to be finitely generated if there is a finite symmetric set $S \subset G$, such that S generates G i.e. $G = \langle S \rangle$. By symmetric set we mean to say S is closed under taking inverse. Here we always take G to be a finitely generated group. A *Cayley graph* of G with respect to S is a graph, denoted by $\Gamma(G, S)$, whose vertex set is G, and between g, h there is an edge if and only if $g^{-1}h \in S$. Naturally the condition $G = \langle S \rangle$ forces $\Gamma(G, S)$ to be connected graph.

Suppose we declare the length of each edge of $\Gamma = \Gamma(G, S)$ is 1. Then for any two vertices $g, h \in \Gamma$, there is a path in Γ joining g to h (because Γ is connected). Take $p_{g,h}$ to be one such path, then its length is given by the number of edges contained in it and is denoted as $\ell(p_{g,h})$. With the help of this length function on paths one can define a metric d_Γ on Γ:

$$d_\Gamma(g, h) := inf\{\ell(p_{g,h}) : p_{g,h} \text{ is a path from } g \text{ to } h\}.$$

Remark 5.1.1 *(Γ, d_Γ) is a locally finite metric space.*

Let us take another finite generating set S' of G, then we also have the metric space $(\Gamma' = \Gamma(G, S'), d_{\Gamma'})$. Therefore we simply have two (different) metrics defined on the same group G, coming from two Cayley graphs. It is natural to ask whether these two metrics are 'comparable' or not. In coarse geometry this "comparability" of metric spaces is provided by *quasi-isometry*.

Definition 5.1.2 *Let (X, d_X) and (Y, d_Y) are two metric spaces. They are called quasi-isometric if there is a function $f : (X, d_X) \to (Y, d_Y)$ and a constant $C > 0$ such that*

1. *$\frac{1}{C} d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq C d_X(x, x') + C$ for all $x, x' \in X$ and*
2. *$\mathcal{N}_C(f(X)) := \{y \in Y : \exists\ x \in X \text{ with } d_Y(y, f(x)) \leq C\} = Y$.*

It is easy to see that for any two finite generating set S and S' of a group G, the metric spaces $(\Gamma(G, S), d_\Gamma)$ and $(\Gamma(G, S'), d_{\Gamma'})$ are quasi-isometric. If we choose three vertices in (Γ, d_Γ), and join any two vertices by a geodesic (length minimizing path), what we have is called a triangle. This triangle is δ-thin if each side is contained in the union of δ-neighborhoods of the other two.

Definition 5.1.3 *A group G is called a hyperbolic group if the Cayley graph of G with respect to a finite generating set S, $\Gamma(G,S)$, is a δ-hyperbolic graph, for some $\delta \geq 0$, where by δ-hyperbolic graph we mean a graph in which every triangle is δ-thin.*

Notice that hyperbolicity does not depend on the choice of generating set, because of the presence of quasi-isometry between any two Cayley graphs with respect to finite generating set. The group $\mathbb{Z}$ is hyperbolic, so is free groups. Any group G acts on its Cayley graph $\Gamma(G,S)$ ($|S| < \infty$), and the action is properly discontinuous, i.e. for every finite set $K \subset \Gamma(G,S)$ the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite. A generalisation of hyperbolic group is relatively hyperbolic group. They are those groups which acts properly on a locally (non-uniformly) finite hyperbolic graph [7, 17].

The notion of relatively hyperbolic groups was further generalized to acylindrically hyperbolic groups. Suppose G is a finitely generated group which acts on a hyperbolic metric space (X,d) via isometry. The action will be called acylindrical if long cylinders in the metric space are fixed by atmost finitely many elements of G. To be precise:

Definition 5.1.4 *[30] An action $G \curvearrowright (X,d)$ is called acylindrical if for every $\epsilon > 0$ there is $R = R(\epsilon) \in \mathbb{R}_{\geq 0}$ and $N = N(R,\epsilon) \in \mathbb{N}$ such that whenever $d(x,y) \geq R$ then the following holds:*

$$\#\{g \in G : gx \in B(x;\epsilon) \ \&\ gy \in B(y;\epsilon)\} \leq N$$

where $B(x;\epsilon)$ signifies the ball in X, about x or radius ϵ. Groups with such kind of action on a hyperbolic metric space is called acylindrically hyperbolic groups.

Hyperbolic groups are acylindrically hyperbolic. Let G be a group hyperbolic relative to collection $\mathcal{H}$, then G acts on $\hat{\Gamma}(G, S \cup \mathcal{H})$ acylindrically. Let Σ_g is a genus g surface ($g \geq 2$). $\mathcal{C}(\Sigma_g)$ denote the curve complex of Σ_g and $MCG(\Sigma_g)$ be the Mapping Class Group of Σ_g. Bowditch proved that $MCG(\Sigma_g)$ acts acylindrically on $\mathcal{C}(\Sigma_g)$ []. $MCG(\Sigma_g)$ is an acylindrically hyperbolic group which is neither hyperbolic itself nor hyperbolic relative to any collection of its subgroups. Again if $\mathbb{F}_n$ is a free group on n generators, then $Out(\mathbb{F}_n)$ it is acylindrically hyperbolic group. Another generalisation of amenability was formulated by R. Zimmer in chapter 4 of [39] and it was called amenable action on some Borel space. Suppose G is an amenable group, then an equivalent way of saying this is the existence of a sequence $\{\theta_n\}$ of probability maps on G, that is asymptotically invariant under the canonical action of G, i.e. for all $g \in G$

$$||g \cdot \theta_n - \theta_n||_1 \to 0 \quad \text{as } n \to \infty.$$

Suppose X is a compact topological (or Borel) space. The probability space over X, $Prob(X)$, which is collection of all measures μ on X such that $\mu(X) = 1$, has the weak-*-topology. Suppose G is a group which acts on X via automorphism, then G has a canonical action on $Prob(X)$, given by

$$g \cdot \mu(x) := \mu(g^{-1}x)$$

when $g \in G$ and $\mu \in Prob(X)$. Like amenability, amenable action also admits several equivalent definition. The one we will be following is:

Definition 5.1.5 *[23] A compact (Borel) G-space X is called amenable or the action of the group is topologically (Borel) amenable if there exist a sequence of continuous (Borel) maps,*

$\mu_n : X \longrightarrow Prob(G)$, *such that for every* $g \in G$ *and for all (almost every)* $x \in X$ *the following holds:*

$$\lim_{n\to\infty} ||g.\mu_n^x - \mu_n^{g.x}||_1 = 0 \qquad (\mu_n^x := \mu_n(x))$$

In particular if X as a singleton, then we see that the definition of amenable action is same as the group G being amenable. In literature whenever G admits a topological amenable action, G is called a group with boundary amenability property.

For the group $\mathbb{F}_2$ the Gromov boundary $\partial\mathbb{F}_2$ is compact and the action is via homeomorphism. Now for any $x \in \partial\mathbb{F}_2$, if we define $\mu_n^x := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{x_k}$, where $\{x_n\}_{n=1}^{\infty}$ is the sequence of consecutive points on the unique geodesic joining 1 to x. Then it is easy to show that for every $g \in \mathbb{F}_2$, we will have $||g \cdot \mu_n^x - \mu_n^{gx}|| \leq \frac{2|g|}{n}$ and hence the action $\mathbb{F}_2 \curvearrowright \partial\mathbb{F}_2$ is amenable.

Adams proved that hyperbolic groups admits a borel amenable action on their Gromov boundary, in the sense of Zimmer [1] (this action is moreover topologically amenable, proved by Germain in the appendix of [3]). Later Kaimanovich generalized this to the case of groups which have exponentially bounded geometry [24].

Later Ozawa in [31], proved that when G is hyperbolic relative to a finite collection of subgroups $\mathcal{H}$, then on the corresponding *Bowditch boundary* $\partial_{\mathcal{B}}(G, \mathcal{H})$ the action of G is topological amenable provided every element in $\mathcal{H}$ is amenable.

Furthermore, some particular examples of groups from the class of acylindrically hyperbolic groups are shown to exhibit amenable actions. These are:

1. The mapping class groups of genus g surface, denoted $MCG(\Sigma_g)$ admits borel amenable action on some borel space [26]. Later this result was strengthened in [22] to topological amenability.

2. The same for $Out(\mathbb{F}_n)$ (the outer automorphism group of finite rank free groups) was proved in [6].

Since $MCG(\Sigma_g)$, $Out(\mathbb{F}_n)$ are typical examples of acylindrically hyperbolic groups which are not relatively hyperbolic, the natural question to ask here is whether all acylindrically hyperbolic group admits topological amenable action on some compact space. But the answer is negative, as the point stabilizers under the action have to be amenable in that case. So what we have proved is the following:

Theorem 5.1.6 *Suppose G is a finitely generated acylindrically hyperbolic group. Then there is a hyperbolic space (not necessary locally finite) on which G acts acylindrically, and the natural action of the group on the boundary of the hyperbolic space (along with a suitable measure) is borel amenable.*

Chapter 6

Future Plan

6.1 Weak amenable groups

We have mentioned earlier what it means for a finitely generated group G to be weakly amenable. A formal definition is:

Definition 6.1.1 *G is weakly amenable if there is a sequence of functions $\phi_n : G \to \mathbb{C}$ satisfying:*

1. *support of each ϕ_n is finite,*
2. *$\phi_n(g) \to 1$ as $n \to \infty$, for all $g \in G$,*
3. *there exists some $C > 0$ such that $||\phi_n||_{cb} \leq C$, $\forall n \in \mathbb{N}$.*

The Cowling-Haagerup constant for G, denoted by $\Lambda_{CH}(G)$, is the infimum of all such $C > 0$ so that the above conditions hold.

All amenable groups are weakly amenable. It was proved in [8] that free product of finitely many amenable groups are weakly amenable. This result was further generalized in [35]. The weak amenability of finite rank free groups follows from [21]. Following the proofs of [21, 8], later Ozawa, in [32], proved that all the hyperbolic groups are weakly amenable. A natural question now is when are relatively hyperbolic groups exhibit weak amenability. Some partial answers that were found till date are:

1. Free products of two weakly amenable groups with $\Lambda_{CH} = 1$ ([35]).
2. Relatively hyperbolic groups where maximal parabolic subgroups have at-most polynomial growth ([20]).
3. Free products of hyperbolic group and amenable group ([38]).

Therefore we are trying to address the following:

Question 6.1.2 *Suppose G is hyperbolic relative to the collection of subgroups $\{H_i\}_{i=1}^n$. If we have $\Lambda_{CH}(H_i) = 1$ for all i, then G is weakly amenable.*

6.2 Boundary amenability

We have showed that acylindrically hyperbolic group admits borel amenable action on some measure space. It was proved that for a finitely generated group G, if there is a *Floyd scaling function* $f : \mathbb{N} \to \mathbb{R}_{>0}$, so that the *Floyd boundary* of G, $\partial_f G$, is non-trivial, then G is acylindrically hyperbolic [37]. The boundary, $\partial_f G$, is a compact topological space on which G acts via homeomorphism.

Whenever G is a hyperbolic group, the Gromov boundary of G, on which it admits amenable action, is Floyd boundary. Similarly if G is relatively hyperbolic with respect to a finite collection of amenable subgroups, then G admits amenable action on the corresponding *Bowditch boundary* of the group, that again coincides with Floyd boundary of G. Therefore we want to know the following:

Question 6.2.1 *If G is a finitely generated group with non-trivial Floyd boundary, $\partial_f G$. Then under which assumption $G \curvearrowright \partial_f G$ is topologically amenable.*

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